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New Directions in Algebraic Dynamical Systems

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Abstract—The logarithmic Mahler measure of certain multivariate polynomials occurs frequently as the entropy or the free energy of solvable lattice models (especially dimer models). It is also known that the entropy of an algebraic dynamical system is the logarithmic Mahler measure of the defining polynomial. The connection between the lattice models and the algebraic dynamical systems is still rather mysterious.

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To Henk Broer, on the occasion of his 60^{th} birthday

1. INTRODUCTION

1.1. Chessboard and Domino Tilings

Consider a chessboard of size $2n \times 2n$. What is the number of ways to cover the board with domino tiles whose size is exactly two board squares? For small n, the numbers of domino tilings of $Z_{2n\times 2n}$ are given by

n	1	2	3	4
$Z_{2n\times 2n}$	2	36	6728	12988816

A general formula was obtained independently by Kasteleyn [1] and Temperley & Fisher [2] in 1961:

$$\begin{aligned} Z_{2n\times 2n} &= \prod_{j=1}^{2n}\prod_{k=1}^{2n}\left(4\cos^2\frac{\pi j}{2n+1}+4\cos^2\frac{\pi k}{2n+1}\right)^{1/4} \\ &= \prod_{j=1}^{2n}\prod_{k=1}^{2n}\left(4+2\cos\frac{2\pi j}{2n+1}+2\cos\frac{2\pi k}{2n+1}\right)^{1/4}. \end{aligned} \tag{1.1}$$

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As the data above suggests, $Z_{2n\times 2n}$ grows rapidly. In fact, $Z_{2n\times 2n}$ is exponential in *area* ($4n^2$), and the growth rate is

$$\begin{aligned}\varkappa = \lim_{n\to\infty}\frac{1}{(2n)^2}\log Z_{2n\times 2n} &= \frac{1}{4}\int_0^1\int_0^1 \log\Big(4+2\cos(2\pi\theta_1)+2\cos(2\pi\theta_2)\Big)\,d\theta_1 d\theta_2\\ &= \frac{1}{4}\int_0^1\int_0^1 \log\Big(4-2\cos(2\pi\theta_1)-2\cos(2\pi\theta_2)\Big)\,d\theta_1 d\theta_2.\end{aligned}\tag{1.2}$$

After taking the logarithm in (1.1), the product becomes a Riemann sum, converging to the integral in (1.2), see [1]; some attention has to be paid to the singularity of

$$\log\big(4+2\cos(2\pi\theta_1)+\cos(2\pi\theta_2)\big)$$

at $\theta_1=\theta_2=\frac{1}{2}$. The value of the double integrals in (1.2) can be computed

$$\int_0^1\int_0^1 \log\Big(4-2\cos(2\pi\theta_1)-\cos(2\pi\theta_2)\Big)d\theta_1 d\theta_2 = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} = \frac{4}{\pi}L(\chi,2) = \frac{4G}{\pi},$$

where $L(\chi,s)$ is the Dirichlet L-function for the non-trivial character modulo 4, and G is the Catalan constant. The appearance of a special value of an L-function is not accidental, as we will discuss below.

The growth rate $\varkappa$ is also the *topological entropy* of the symbolic dynamical system obtained by considering all infinite domino tilings of $\mathbb{Z}^2$. For comparison with other systems discussed below, it is convenient to get rid of the factor 1/4 in (1.2). This can be achieved by considering the *even shift* action of $\mathbb{Z}^2$ on the set of infinite domino tilings, i.e., shifts by $2\mathbf{n}$, $\mathbf{n}\in\mathbb{Z}^2$. The topological entropy of this action is $4\varkappa$.

Finally, it is well known that domino tilings of $\mathbb{Z}^2$ are in one-to-one correspondence with dimer matchings, see figure 1.

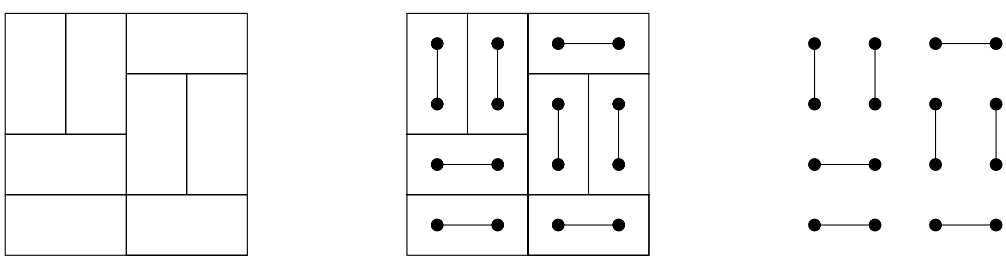

Fig. 1. Correspondence between domino tilings and dimer matchings in $\mathbb{Z}^2$.

In this sense, domino tilings and dimer matchings are synonymous, and we will use both terms interchangeably.

1.2. Mahler Measure

For a Laurent polynomial $f(u_1,\ldots,u_d)$ in d-variables, the logarithmic Mahler measure is given by

$$\mathsf{m}_f = \int_0^1\cdots\int_0^1 \log\left|f\left(e^{2\pi i\theta_1},\ldots,e^{2\pi i\theta_d}\right)\right|\,d\theta_1\cdots d\theta_d.$$

For a nonzero polynomial f with integer coefficients, the logarithmic Mahler measure m_f is non-negative. The logarithmic Mahler measure satisfies

$$\mathsf{m}_{f\cdot g} = \mathsf{m}_f + \mathsf{m}_g.$$

For a nonzero polynomial f depending only on one variable, we have $f(z) = a\prod_{i=1}^{n}(z - z_i)$, for some $a, z_i \in \mathbb{C}$, and Jensen's formula provides a closed form expression for the logarithmic Mahler measure

$$\mathsf{m}_f = \log|a| + \sum_{i:|z_i|>1} \log|z_i|.$$

The growth rate of the number of domino tilings in (1.2) is given by $\varkappa = \frac{1}{4}\mathsf{m}_f$, where

$$f = 4 - (u_1 + u_1^{-1} + u_2 + u_2^{-1}).$$

The special values of L-functions often appear as the values of logarithmic Mahler measures. For example,

- (Smyth, [3])

$$\mathsf{m}_{1+u_1+u_2} = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1), \quad \text{where} \quad L(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s}$$

  is the Dirichlet L-series for the non-principal character modulo 3 χ_{-3}: $\chi_{-3}(n) = k \in \{-1, 0, 1\}$ if and only if $n \equiv k \mod 3$.

- (Smyth, [4])

$$\mathsf{m}_{1+u_1+u_2+u_3} = \frac{7}{2\pi^2}\zeta(3),$$

  where ζ is the Riemann zeta-function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1.$$

Any Mahler measure is a basic example of a *period* introduced by Kontsevich & Zagier [5]. In general, the fact that special values of L-functions can be expressed as periods is very intriguing. The general program outlined in [5] has attracted a lot of attention. For Mahler measures of polynomials, Deninger [6] established that under suitable conditions the Mahler measure is a Deligne period of a mixed motive.

One particular family of polynomials (which is also relevant for models of Statistical Mechanics discussed below) has been studied rather extensively:

$$f_k = k - (u_1 + u_1^{-1} + u_2 + u_2^{-1}), \quad k \in \mathbb{Z},$$

Boyd [7] verified numerically (to a very high degree of accuracy) that for $1 \leqslant k \leqslant 100$, $k \neq 4$, one has

$$\mathsf{m}_{f_k} = r_k L'(E_k, 0), \tag{1.3}$$

where $r_k \in \mathbb{Q}$, E_k is the elliptic curve corresponding to the null set $\{f_k = 0\}$, and L is the corresponding L-function. Deninger [6] also related the logarithmic Mahler measure m_{f_k} to Eisenstein–Kronecker series. His result, together with the Bloch-Beilinson conjectures, implies (1.3). Rodriguez–Villegas [8] developed alternative approaches to the evaluation of m_{f_k}.

1.3. Algebraic Dynamical Systems

Suppose f is a Laurent polynomial in d-variables ($f \in R_d := \mathbb{Z}[u_1, \dots, u_d, u_1^{-1}, \dots, u_d^{-1}]$). Consider the following subgroup of $\mathbb{T}^{\mathbb{Z}^d}$:

$$X_f = \left\{ (x_{\mathbf{n}}) \in \mathbb{T}^{\mathbb{Z}^d} : \sum_{\mathbf{m} \in \mathbb{Z}^d} f_{\mathbf{m}} x_{\mathbf{m}+\mathbf{n}} = 0 \text{ for all } \mathbf{n} \in \mathbb{Z}^d \right\}. \tag{1.4}$$

Let α_f be the restriction to X_f of the natural shift-action of $\mathbb{Z}^d$ on $\mathbb{T}^{\mathbb{Z}^d}$; α_f is a $\mathbb{Z}^d$-action by automorphisms of the compact group X_f. The dynamical system (X_f, α_f) is an example of an *algebraic dynamical system*: namely, a dynamical system with a phase-space *group* and dynamics given by automorphisms of that group. The algebraic $\mathbb{Z}^d$-action α_f on X_f is completely determined by f, and one can express its dynamical properties in terms of the Laurent polynomial f as follows.

(a) X_f is infinite if and only if f is not a unit in R_d, i.e., if and only if f is not of the form $\pm u^{\mathbf{n}}$ for some $\mathbf{n} \in \mathbb{Z}^d$;

(b) X_f is connected if and only if f is *primitive*, i.e., not divisible by an integer $m > 1$;

(c) α_f is mixing (with respect to the normalized Haar measure λ_{X_f} of X_f) if and only if f is not divisible by a polynomial of the form $c(\mathbf{n})$, where $\mathbf{0} \neq \mathbf{n} \in \mathbb{Z}^d$ and $c \in R_1$ is a cyclotomic polynomial;

(d) If α_f is mixing it has positive entropy and is isomorphic to a Bernoulli shift;

(e) α_f is expansive if and only if

$$V(f) = \left\{ \mathbf{c} = (c_1, \dots, c_d) \in (\mathbb{C} \setminus \{0\})^d : f(\mathbf{c}) = 0 \right\} \cap \mathbb{S}^d = \varnothing, \tag{1.5}$$

where $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$. By a theorem of Lind, Schmidt, and Ward [9], the topological entropy of (X_f, α_f) is given by the logarithmic Mahler measure of f:

$$h_{\mathsf{top}}(\alpha_f) = \mathsf{m}_f.$$

An immediate question is whether there is any relation between the dimer model and the algebraic dynamical system X_f corresponding to the *Laplacian* $f = 4 - (u_1 + u_1^{-1} + u_2 + u_2^{-1})$. The question was raised by Burton & Pemantle [10] in 1993. Indeed, in measurable sense the systems are isomorphic. Since the even shift of the dimer model and the algebraic dynamical system X_f have unique measures of maximal entropy with equal entropies and both systems are Bernoulli, they are isomorphic in the measure-theoretic category: i.e., there are sets of full measures and a measure preserving bijective map between those sets commuting with the shift actions. However, such an answer is too weak. There are good reasons to expect a stronger relations between the systems. Moreover, as we will see below, the coincidence of entropies, and more general, of free energies, is not exceptional, and holds for similar systems as well.

1.4. Solvable Models of Statistical Mechanics

The Dimer model discussed above is an example of a *solvable model* in Statistical Mechanics. This means that the *solutions* can be expressed explicitly in terms of some previously known functions. This notion is best demonstrated on the basis of examples.

Consider a model of Statistical Mechanics given by a formal Hamiltonian (energy) H

$$H(\sigma) = \sum_{\Lambda} U_{\Lambda}(\sigma_{\Lambda}),$$

with the local terms U_Λ determining the energy of a finite subconfiguration σ_Λ. For example, for the Ising model

$$H(\sigma) = -J \sum_{i \sim j} \sigma_i \sigma_j,$$

where the sum is taken over all nearest neighbors $i \sim j$. The associated *partition function* is given by

$$Z = \sum_{\sigma} e^{-\beta H(\sigma)},$$

where the parameter β is the inverse temperature of the system. In the Gibbs formalism, the probability that the system is in a state σ is given by

$$\mathbb{P}(\sigma) = \frac{1}{Z} e^{-\beta H(\sigma)}.$$

Finally, the *free energy* is obtained as a thermodynamic limit (i.e., the limit as the number of particles tends to infinity)

$$F_\beta = \lim_{n\to\infty} -\frac{1}{\beta n} \log Z.$$

In some cases, the free energy can be determined explicitly. For example, the combinatorial result of Kasteleyn is in fact determining the free energy of a dimer model with $H \equiv 0$. Remarkably, in many known cases, the free energy turns out to coincide with the logarithmic Mahler measure of a certain polynomial. Below we list some of the models, their resulting free energies F, and the corresponding polynomials f. For simplicity we take $\beta = 1$.

Dimer model [1] $\quad F = -\frac{1}{4}\mathsf{m}_f, \quad f = 4 - (u_1 + u_1^{-1} + u_2 + u_2^{-1})$

2D Ising model [11] $\quad F = -\frac{1}{2}\mathsf{m}_f, \quad f = 4\Big(\frac{1+T^2}{1-T^2}\Big)^2 - \frac{4T}{1-T^2}(u_1 + u_1^{-1} + u_2 + u_2^{-1}),$ where $T = \tanh(J)$

Conjugate model [12] $\quad F = -\frac{1}{2}\mathsf{m}_f, \quad f = a - b(u_1u_2^{-1} + u_1^{-1}u_2) - c(u_1u_2 + u_1^{-1}u_2^{-1})$

Free-fermion model [12] $\quad F = -\frac{1}{2}\mathsf{m}_f, \quad f = a - b(u_1 + u_1^{-1}) - c(u_2 + u_2^{-1}) - d(u_1u_2^{-1} + u_1^{-1}u_2) - e(u_1u_2 + u_1^{-1}u_2^{-1})$

1.5. No Coincidences!

A natural question is whether there is a relation between a solvable model whose solution is given by a logarithmic Mahler measure of a certain polynomial, and the algebraic dynamical system corresponding to that polynomial. The problem remains open even in the case of the dimer model. In the next section we describe the first step towards the affirmative solution. Namely, we describe the correspondence between the algebraic dynamical systems corresponding to

$$f = k - (u_1 + u_1^{-1} + u_2 + u_2^{-1}), \quad k \geqslant 4,$$

and the so-called Abelian Sandpile Models (ASM), defined below. The critical Abelian Sandpile Model ($k = 4$) is supposed to be strongly related to the planar dimer model (this question was raised e.g. in [13]). The reason is that in *finite volumes* there is indeed a strong correspondence between various models, where the link is given by *spanning trees* [10]. Here is the summary of known results on one-to-one correspondences between various models in finite volume:

- [14]: there are as many dimer matchings on a square box $[1, 2n-1] \times [1, 2n-1]$ without a corner as there are spanning trees on the square box $[1, n] \times [1, n]$;
- [15]: there as many spanning trees on the square box $[1, n] \times [1, n]$ plus one additional vertex, connected to all boundary sites, as there are ASM configurations on a square box $[1, n] \times [1, n]$;
- [16]: there as many spanning trees on the square box $[1, n] \times [1, n]$ plus one additional vertex, connected to all boundary sites, as there are elements of a particular finite subgroup $X_f^{(n)}$ of $\mathbb{T}^{[1,n]^2}$, with

$$\lim_{n\to\infty} \frac{1}{n^2} \log \left| X_f^{(n)} \right| = \mathsf{m}_f.$$

More generally, a plausible hypothesis is that various dimer models could form symbolic covers of corresponding algebraic dynamical systems. In the last section we elaborate further on this idea. Finally, though Onsager's original solution of 2D Ising model [11] did not involve dimers, subsequent work showed that one can represent the Ising model as a dimer model on a decorated lattice [17]. Since dimer models appear behind many solvable models of Statistical Mechanics, it is rather natural to address the correspondence between the dimer and algebraic systems first. In the last section, we also outline one alternative approach which could link the 2D Ising model and the corresponding algebraic system directly.

2. ABELIAN SANDPILE MODEL AND THE HARMONIC MODELS

The d-dimensional ***Abelian sandpile model*** was introduced by Bak, Tang and Wiesenfeld in [18, 19] and attracted a lot of attention after the discovery of the Abelian property by Dhar in [15]. For a detailed mathematical introduction into the model see [20].

2.1. Abelian Sandpile Models

Let us view the lattice $\mathbb{Z}^2$ as an infinite graph, where nearest neighbors are connected by an edge. Consider a finite subgraph Λ of $\mathbb{Z}^2$. By definition, *configurations* are elements of $\mathbb{N}^\Lambda = \{\eta : \Lambda \to \mathbb{N}\}$. A configuration η is *stable* if $\eta_i = \eta(i) \leqslant 4$ for all $i \in \Lambda$. Configurations which are not stable, can be stabilized by means of repeated application of *toppling operators*. For every $i \in \Lambda$, the toppling operator T_i only acts on configurations η such that $\eta_i > 4$. For such configurations define $\eta' = T_i(\eta)$ as follows: for every $j \in \Lambda$,

$$\eta'_j = \begin{cases} \eta_i - 4, & \text{if } j = i, \\ \eta_j + 1, & \text{if } i, j \text{ are nearest neighbors in } \mathbb{Z}^2, \\ \eta_j, & \text{otherwise.} \end{cases}$$

The action of toppling operators can be visualized as follows: if the site is unstable, then the site topples by giving one grain of sand to each of the nearest neighbors. For boundary sites – sites in Λ with fewer than 4 nearest neighbors in Λ, some grains of sand are lost. If we view configurations as column vectors in $\mathbb{N}^\Lambda$, then

$$\eta' = T_i(\eta) = \eta - \Delta_\Lambda \boldsymbol{\delta}^{(i)}, \tag{2.1}$$

where $\boldsymbol{\delta}_j^{(i)} = \delta_{ij}$, $j \in \Lambda$, and Δ_Λ is square matrix of size $|\Lambda|$ given by

$$\Delta_{ij} = \begin{cases} 4, & \text{if } i = j, \\ -1, & \text{if } ||i - j|| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

In fact, Δ_Λ is the so-called *graph Laplacian* of Λ. From the representation (2.1) of toppling operators one immediately concludes that for all $i, j \in \Lambda$, the toppling operators T_i, T_j *commute*:

$$T_i \circ T_j = T_j \circ T_i. \tag{2.2}$$

Given an unstable configuration η, we keep applying the toppling operators at each unstable site, until a stable configuration is reached. In view of (2.2), the order of topplings is irrelevant. The process will eventually stop, because of the dissipativity of topplings on the boundary. We denote by $\mathcal{T}(\eta)$ the result of stabilization of η.

Given two configurations $\eta, \rho \in \mathbb{N}^\Lambda$, define the addition operation $\eta \oplus \rho$ as the result of stabilization of $\eta + \rho$ (added coordinatewise):

$$\eta \oplus \rho = \mathcal{T}(\eta + \rho).$$

We say that a stable configuration η is *recurrent* if

$$\eta = \eta \oplus \rho = \mathcal{T}(\eta + \rho)$$

for some configuration ρ. It turns out that the set of recurrent configuration $\mathcal{R}_\Lambda$ forms a group with a group operation $\oplus$. The group $(\mathcal{R}_\Lambda, \oplus)$ is called the *sandpile group* of Λ. Moreover,

$$\mathcal{R}_\Lambda \cong \mathbb{Z}^\Lambda / \Delta_\Lambda \mathbb{Z}^\Lambda,$$

and hence $|\mathcal{R}_\Lambda| = \det(\Delta_\Lambda)$.

The above construction is easily generalized to higher dimensions: 4 has to be substituted by $2d$ – the number of nearest neighbors in $\mathbb{Z}^d$. Another interesting generalization is the class of *dissipative* sandpile models. These models are parametrized by an integer parameter $\gamma > 0$. A configuration η is now called stable if $\eta_i \leqslant 2d + \gamma$, and every time a site topples, $2d + \gamma$ grains of sand are removed, but each nearest neighbor still gets only 1 grain. In this way, sand is lost at every toppling, and not only on the boundary. We will denote by $\mathcal{R}_\Lambda^{(d,\gamma)}$ the set of recurrent configurations of the Abelian sandpile model with parameter $\gamma \geqslant 0$ on a finite $\Lambda \subset \mathbb{Z}^d$.

Finally, we define the set of infinite recurrent configurations $\mathcal{R}_\infty^{(d,\gamma)}$. This set is formed by all stable configurations on $\mathbb{Z}^d$ with the property that the restriction to any finite volume Λ is recurrent, i.e., is an element of $\mathcal{R}_\Lambda^{(d,\gamma)}$:

$$\mathcal{R}_\infty^{(d,\gamma)} = \left\{ \eta \in \{1, 2, \ldots, 2d + \gamma\}^{\mathbb{Z}^d} : \quad \eta|_\Lambda \in \mathcal{R}_\Lambda^{(d,\gamma)} \text{ for all finite } \Lambda \subset \mathbb{Z}^d \right\}.$$

The set $\mathcal{R}_\infty^{(d,\gamma)}$ is a subshift (closed, shift invariant subset) of the full shift space $\{1, 2, \ldots, 2d + \gamma\}^{\mathbb{Z}^d}$. It is easy to see that $\mathcal{R}_\infty^{(d,\gamma)}$ is not a subshift of finite type. The properties of $\mathcal{R}_\infty^{(d,\gamma)}$ as a symbolic dynamical system are not well understood. Nevertheless, the topological entropy of the$\mathbb{Z}^d$-shift action on $\mathcal{R}_\infty^{(d,\gamma)}$ can be shown to coincide with the logarithmic Mahler measure of

$$f_k^{(d)} = k - \sum_{i=1}^{d} (u_i + u_i^{-1}), \quad k = 2d + \gamma. \tag{2.3}$$

2.2. Symbolic Covers and Homoclinic Points

Let $(X_{f_k^{(d)}}, \alpha_{f_k^{(d)}})$ be the algebraic dynamical system corresponding to the Laurent polynomial $f_k^{(d)}$ given by (2.3). Dynamically, there is an important difference between the systems corresponding to $k = 2d$ (critical case) and $k > 2d$ (dissipative case). In the critical case, dynamical system $(X_{f_k^{(d)}}, \alpha_{f_k^{(d)}})$ is not expansive, while it is expansive in the dissipative case.

Suppose f is a Laurent polynomial. A subshift $Y \subset A^{\mathbb{Z}^d}$, where $A = \{1, \ldots, N\}$ is a finite set (the *alphabet*), is a called a *symbolic cover* of X_f if

- the shift action σ of $\mathbb{Z}^d$ on Y has the same topological entropy

$$h_{\mathsf{top}}(Y, \sigma) = h_{\mathsf{top}}(X_f, \alpha_f) = \mathsf{m}_f,$$

- there exists a surjective continuous map $\xi : Y \to X_f$ such that for every $\mathbf{n} \in \mathbb{Z}^d$ the following diagram commutes

$$\begin{array}{ccc} Y & \xrightarrow{\sigma^{\mathbf{n}}} & Y \\ \xi \big\downarrow & & \big\downarrow \xi \\ X_f & \xrightarrow{\alpha_f^{\mathbf{n}}} & X_f \end{array}$$

The construction of symbolic covers gives rise to some very interesting arithmetical problems which are still unresolved (cf. [21–24]). The most successful approach to the construction of symbolic covers is based on utilization of *homoclinic points*, where a point $x \in X_f$ is called *homoclinic* if $\alpha_f^{\mathbf{n}} x \to 0$ as $||\mathbf{n}|| \to \infty$, $\mathbf{n} \in \mathbb{Z}^d$. Depending on the properties of the polynomial f, convergence to 0 for various homoclinic points can be arbitrary slow. For our purposes, we need rather rapid convergence; so write

$$\Delta^{(1)}_{\alpha_f}(X_f) = \Big\{ x \in X_f \text{ is homoclinic and } \sum_{\mathbf{n} \in \mathbb{Z}^d} |x_{\mathbf{n}}| < \infty \Big\}.$$

For expansive actions α_f (i.e, f with $V(f) = \varnothing$, c.f., (1.5)), the set $\Delta^{(1)}_{\alpha_f}(X_f)$ of absolutely summable homoclinic points is non-empty [25]. For non-expansive actions, under very mild conditions, absolutely summable homoclinic points also exist [26–28]. These conditions are satisfied by the polynomials $f_k^{(d)}$ with $k \geqslant 2d$ above (c.f., (2.3)).

More specifically, one can construct homoclinic points as follows. Consider solutions of the *fundamental equation* $f \cdot w = \delta_0$, i.e., for every $\mathbf{n} \in \mathbb{Z}^d$

$$(f \cdot w)_{\mathbf{n}} := \sum_{\mathbf{m} \in \mathbb{Z}^d} f_{\mathbf{m}} w_{\mathbf{m}+\mathbf{n}} = \begin{cases} 1, & \textit{if } \mathbf{n} = \mathbf{0}, \\ 0, & \text{otherwise}. \end{cases}$$

Here, $w : \mathbb{Z}^d \to \mathbb{R}$. If $V(f) = \varnothing$, then a solution $w = (w_{\mathbf{n}})$ can be found as Fourier transforms of $1/f$:

$$w_{\mathbf{n}} = \int_{\mathbb{T}^d} \frac{1}{\sum_{\mathbf{m}} f_{\mathbf{m}} e^{-2\pi i \langle \mathbf{m}, \boldsymbol{\theta} \rangle}} e^{-2\pi i \langle \mathbf{n}, \boldsymbol{\theta} \rangle} d\boldsymbol{\theta} \tag{2.4}$$

Note that since the denominator is never zero, $1/f$ is a smooth function on $\mathbb{S}^d$, and $|w_{\mathbf{n}}|$ decays exponentially fast as $||\mathbf{n}|| \to \infty$. If $V(f) \neq \varnothing$, (2.4) still makes sense, but typically, $w = (w_{\mathbf{n}})$ is not absolutely summable. For example, for the 2D-Laplacian

$$f = 4 - (u_1 + u_1^{-1} + u_2 + u_2^{-1}),$$

$$|w_{\mathbf{n}}| = \mathcal{O}(\log ||\mathbf{n}||) \quad \text{as} \quad ||\mathbf{n}|| \to \infty.$$

Nevertheless, there are always polynomials $g \in \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$, such that the point $v = (v_{\mathbf{n}})$ with

$$v_{\mathbf{n}} = (g \cdot w)_{\mathbf{n}} = \sum_{\mathbf{m}} g_{\mathbf{m}} w_{\mathbf{n}+\mathbf{m}}, \tag{2.5}$$

has absolutely summable coordinates. Clearly, $g = f$ is one such polynomial: $(f \cdot w)_{\mathbf{n}} = \delta_{\mathbf{n},\mathbf{0}}$ for all $\mathbf{n}$. The set of all polynomials g such that $v = g \cdot w \in \ell^1$ is an ideal; denote this ideal by $\mathcal{I}_f$. An important question is when this ideal is different from the principal ideal

$$\langle f \rangle = \{ g = fh, \ h \in \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}] \}.$$

The result of [27], generalizing the previous results [25, 26], states that this is indeed the case for f with finite $V(f)$. In [28], we hope to provide necessary and sufficient conditions for $\mathcal{I}_f \neq \langle f \rangle$ in case $V(f)$ is infinite.

Suppose now that $g \in \mathcal{I}_f$, and let $v = g \cdot w$ be given by (2.5). Suppose that $\eta : \mathbb{Z}^d \to \mathbb{Z}$ is bounded: $||\eta||_\infty = \sup_{\mathbf{n}} |\eta_{\mathbf{n}}| < \infty$. Then, since v is absolutely summable, for any $\mathbf{n}$, the series

$$\sum_{\mathbf{m}\in\mathbb{Z}^d} v_{\mathbf{m}}\eta_{\mathbf{m}+\mathbf{n}}$$

converges. Finally, define $\xi_g(\eta) = (\xi_g(\eta)_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^d} \in \mathbb{T}^{\mathbb{Z}^d}$ as follows:

$$\xi_g(\eta)_{\mathbf{n}} = \sum_{\mathbf{m}\in\mathbb{Z}^d} v_{\mathbf{m}}\eta_{\mathbf{m}+\mathbf{n}} \pmod 1, \quad \mathbf{n} \in \mathbb{Z}^d. \tag{2.6}$$

It is not very difficult to see that in fact $\xi_g(\eta) \in X_f$; moreover, the map $\xi_g : \ell^\infty(\mathbb{Z}^d, \mathbb{Z}) \longmapsto X_f$ is surjective. It is also evident that ξ_g is continuous in the product topology and is equivariant:

$$\alpha_f^{\mathbf{n}} \circ \xi_g = \xi_g \circ \sigma^{\mathbf{n}}.$$

We now summarize the main result of [26]:

Theorem 2.1. *Let $d \geqslant 2$ and*

$$f_k^{(d)} = k - \sum_{i=1}^{d}(u_i + u_i^{-1}), \quad k = 2d + \gamma,\ \gamma \in \mathbb{Z}_+. \tag{2.7}$$

(a) For $k > 2d$ (the dissipative case), $\mathcal{I}_{f_k^{(d)}} = \mathbb{Z}[u_1^{\pm1}, \ldots, u_d^{\pm1}]$, and for $k = 2d$ (the critical case)

$$\mathcal{I}_{f_{2d}^{(d)}} = \langle f_{2d}^{(d)} \rangle + \mathcal{J}_d^3, \quad \mathcal{J}_d = \big\{h : h(1, 1, \ldots, 1) = 0\big\}.$$

In both cases, $\langle f_k^{(d)} \rangle \subsetneq \mathcal{I}_{f_k^{(d)}}$.

(b) Suppose $g \in \mathcal{I}_{f_k^{(d)}} \setminus \langle f_{2d}^{(d)} \rangle$. Then $\xi_g(\mathcal{R}_\infty^{(d,\gamma)}) = X_{f_k^{(d)}}$.

As an immediate consequence of Theorem (2.1), we conclude that $\mathcal{R}_\infty^{(d,\gamma)}$ is a symbolic cover of $X_{f_k^{(d)}}$.

An interesting question which remained unanswered in [26] is whether $\mathcal{R}_\infty^{(d,\gamma)}$ admits a unique measure of maximal entropy. It is known that this is the case for $X_{f_k^{(d)}}$ [9]. Based on this fact, one can establish the uniqueness result for the sandpile model, provided the kernel

$$\ker \xi_g = \Big\{\eta \in \mathcal{R}_\infty^{(d,\gamma)} : \quad \exists \eta' \in \mathcal{R}_\infty^{(d,\gamma)},\ \eta' \neq \eta, \quad \text{with } \ \xi_g(\eta') = \xi_g(\eta)\Big\}.$$

is not too large. In the dissipative case ($k > 2d$), the kernel is indeed small, and uniqueness for the measure with maximal entropy follows [26]*Theorem 6.6. In the critical case ($k = 2d$), the problem remains open.

3. NEW DIRECTIONS

3.1. Symbolic Covers of Algebraic Dynamical Systems

As we demonstrated above, Abelian Sandpile Models form symbolic covers of suitable algebraic dynamical systems. It is tempting to conjecture that the planar dimer model is also a symbolic cover of the Harmonic Model. In fact, if that were the case, the dimer model would be a *better* symbolic cover than the critical Abelian Sandpile Model: the set of infinite dimer matchings of $\mathbb{Z}^2$ is a subshift of finite type, while this is not the case for the set of all infinite recurrent configurations of the ASM. One should be able to extend to the dimer model, the method based on coding via the homoclinic points described above. Most probably, the method should not be applied to the dimer model directly, but to an equivalent representation of dimer configurations by means of the so-called *height functions*. Moreover, if successful, other planar dimer models (e.g., on the honeycomb lattice) could be treated in a similar fashion. Recent papers [27, 28] allow construction of homoclinic points for large classes of characteristic polynomials.

3.2. Algebraic Dynamical Systems and Algebraic Geometry

Mahler measure is the natural characteristic of the algebraic dynamical systems corresponding to Laurent polynomials. The appearance of special values of L-functions as entropies of such systems is still rather mysterious. For example, previous works [6–8] on the Mahler measure of $f_k = k - (u_1 + u_1^{-1} + u_2 - u_2^{-1})$ did not establish any connection between curves and algebraic dynamical systems corresponding to f_k. On the other hand, for planar dimer models, some interesting connections to real algebraic geometry have been found [29–31]. We dare to suggest that one should also find a rather strong relation between the dissipative Abelian sandpile model and the characteristic elliptic curves. The successful co-homological approach to dimer models developed in [29–31], is also applicable to sandpile models. In fact, the sandpile groups of finite graphs are even more suitable: such groups are immediately presented as Picard groups, and are also often referred to as graph Jacobians [32]. Recent works, e.g. [33], expose deep analogies between graph Jacobians and Riemann surfaces. Finally, in [34], graphs were presented with sandpile groups which are combinatorially related to elliptic curves over finite fields.

3.3. Alternative Approaches

In [35] it was shown that for a countable number of parameter values in the Ising model

$$H = J \sum_{i \sim j} \sigma_i \sigma_j + h \sum_i \sigma_i,$$

the corresponding Gibbs state can be obtained as a continuous factor of a measure of maximal entropy of a certain subshift of finite type Σ. It is interesting to understand whether in $d = 2$ for $h = 0$, (solvable case), this subshift Σ could form a symbolic cover of the algebraic dynamical system X_f corresponding[1)] to

$$f = 4a - 4b(u_1 + u_1^{-1} + u_2 + u_2^{-1}), \quad a, b \in \mathbb{Q}.$$

If successful, this approach could allow treatment of the two-dimensional Ising model directly, without the need to switch to an equivalent dimer model first.

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[1)] If one considers Laurent polynomials with rational (rather than integral) coefficients, the definition of the associated algebraic dynamical system has to be extended slightly: for $f \in \mathbb{Q}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ the corresponding group X_f is a shift-invariant subgroup of $\mathbb{K}^{\mathbb{Z}^d}$, where $\mathbb{K}$ is a quotient of the solenoid $\widehat{\mathbb{Q}}$.

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